The Diffraction of Electromagnetic Waves by Dielectric Steps in Waveguides

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Abstract—The problem of determining the scattered electromagnetic fields when a dielectric step discontinuity is placed in a waveguide is considered. Although an exact method of solution is not presently known, the recently introduced modified residue-calculus technique (MRCT) can be successfully extended to obtain a very accurate and numerically efficient approximate solution of the semi-infinite dielectric step. A still further extension of the modified residue-calculus method yields the approximate solution for the case of a finite dielectric step. A unique advantage of the present methods is that the degree of accuracy obtained is independent of the relative permittivity of the dielectric material and of the frequency. Thus very high permittivities or frequencies can be considered without an attendant increase in computational complexity. Numerical data are presented which confirm the accuracy of the method.

INTRODUCTION

WITH the increasing use of new dielectric and semiconductor materials it is relevant to reconsider the scattering problem posed by placing dielectric steps in waveguides. Previous methods, which have achieved an approximate solution to this problem, have employed modifications of the variational technique, a quasi-static solution of a singular integral equation, or inversion of the coefficient matrix. Each one of these methods necessarily makes approximations which limits the accuracy or applicability of the particular method. The approach reported here is to extend the recently introduced modified residue-calculus technique (MRCT) [1] to obtain a solution for dielectric steps which is not based on the approximations used in previous methods, and hence achieves increased accuracy as well as numerical efficiency.

The MRCT is an extension of the conventional residue-calculus technique, and is a rapidly convergent numerical method for solving a class of infinite sets of simultaneous linear equations. The principal advantage of the MRCT is that for a certain class of problems it is possible to extract the asymptotic solution exactly. With this knowledge of the asymptotic solution, the original problem is transformed into one that lends itself to efficient numerical solution. This results in a reduction in matrix size by a factor of ten or more when compared to the conventional mode-matching method.

In its original form, the MRCT can be applied to geometries which are simple modifications of the Wei-

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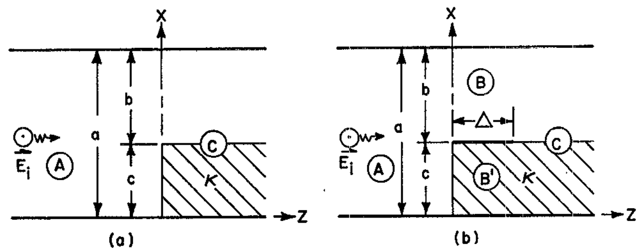

Fig. 1. The semi-infinite dielectric step with auxiliary geometry.

ner–Hopf geometry [2]. This paper reports extensions of the MRCT which allow the solution of a broader class of infinite equations. In particular, the equations obtained in the formulation of the semi-infinite and finite dielectric steps in waveguides are solved by extending the original MRCT.

FORMULATION OF THE SEMI-INFINITE-STEP PROBLEM

A direct mode-matching formulation of the boundary-value problem posed by the semi-infinite dielectric step in an infinite parallel-plate waveguide [Fig. 1(a)] yields a set of infinite equations which presently can be solved only by direct inversion or by an iterative procedure. The size of the matrix required by these methods to obtain sufficient accuracy is an order of magnitude greater than that required if the auxiliary geometry of Fig. 1(b) is used to obtain sets of equations which can be solved by extending the MRCT.

The auxiliary geometry is obtained by placing an infinitely thin, perfectly conducting strip on top of the dielectric at $x=c$ in Fig. 1(a). The strip is infinite in extent in the y direction and extends into the loaded guide a short distance, $z=\Delta$. Note that as $\Delta\to 0$ the auxiliary geometry approaches that of the original problem.

For incident TE_{p0} modes the total electromagnetic fields can be derived from the only nonzero component of electric field E_y

$$H_x = \frac{1}{j\omega\mu_0}\frac{\partial}{\partial z}E_y$$

$$H_z = \frac{-1}{j\omega\mu_0}\frac{\partial}{\partial x}E_y.$$

(exp $(j\omega t)$ time variation is assumed throughout). The transverse electric fields in each of the four regions of the auxiliary geometry are known and may be written as

Region A:

$$E_y = A \sin(p\pi x/a) \exp(-\alpha_p z) + \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \exp(\alpha_n z). \quad (1a)$$

Region B:

$$E_y = \sum_{n=1}^{\infty} \sin[n\pi(x-c)/b] \cdot \{B_n \exp(-\beta_n z) + E_n \exp[\beta_n(z-\Delta)]\}. \quad (1b)$$

Region B′:

$$E_y = \sum_{n=1}^{\infty} \sin(n\pi x/c) \cdot \{\overline{B}_n \exp(-\zeta_n z) + \overline{E}_n \exp[\zeta_n(z-\Delta)]\}. \quad (1c)$$

Region C:

$$E_y = \sum_{n=1}^{\infty} C_n \psi_n \exp[-\gamma_n(z-\Delta)] \quad (1d)$$

where

$$\psi_n = \begin{cases} \sin(x\sqrt{\gamma_n^2 + \kappa k_0^2}), & 0 < x < c \\ \dfrac{\sin(c\sqrt{\gamma_n^2+\kappa k_0^2})}{\sin(b\sqrt{\gamma_n^2+k_0^2})} \sin[(a-x)\sqrt{\gamma_n^2+k_0^2}], & c < x < a \end{cases}$$

and α_n, β_n, ζ_n, γ_n, and k_0 are the wave numbers in regions A, B, B', C, and free space, respectively [3]. Except for a constant factor, the transverse magnetic field in each region is found by partial differentiation with respect to z.

The transverse field components are matched at $z=0$ and $z=\Delta$. Fourier analyzing the resulting equations, letting $\Delta \to 0$, and eliminating the coefficients B_n, $\overline{B}_n$, E_n, and $\overline{E}_n$, the following expressions for the unknown mode coefficients A_n and C_n are obtained:

$$\sum_{n=1}^{\infty}\left[A_n'\left(\frac{1}{\alpha_n-\beta_m'}+\frac{\lambda_m}{\alpha_n+\beta_m'}\right)+\xi_m\frac{C_n'}{\gamma_n+\zeta_m}\right] = A'\left(\frac{1}{\alpha_p+\beta_m'}+\frac{\lambda_m}{\alpha_p-\beta_m'}\right) \quad (2a)$$

$$\sum_{n=1}^{\infty}\left[A_n'\left(\frac{1}{\alpha_n+\beta_m'}+\frac{\lambda_m}{\alpha_n-\beta_m'}\right)+\xi_m\frac{C_n'}{\gamma_n-\zeta_m}\right] = A'\left(\frac{1}{\alpha_p-\beta_m'}+\frac{\lambda_m}{\alpha_p+\beta_m'}\right) \quad (2b)$$

$$\sum_{n=1}^{\infty}\left(\frac{A_n'}{\alpha_n-\beta_m}+\frac{C_n'}{\gamma_n+\beta_m}\right) = \frac{A'}{\alpha_p+\beta_m} \quad (2c)$$

$$\sum_{n=1}^{\infty}\left(\frac{A_n'}{\alpha_n+\beta_m}+\frac{C_n'}{\gamma_n-\beta_m}\right) = \frac{A'}{\alpha_p-\beta_m} \quad (2d)$$

where, to simplify the above expressions, the following definitions have been made:

$$\beta_m' = \sqrt{\zeta_m^2 + (\kappa-1)k_0^2} = \sqrt{(m\pi/c)^2 - k_0^2}$$

$$\lambda_m = \frac{\beta_m' - \zeta_m}{\beta_m' + \zeta_m}$$

$$\xi_m = \frac{2\beta_m'}{\beta_m' + \zeta_m}$$

$A' = A \sin(p\pi c/a)$, $A_n' = A_n \sin(n\pi c/a)$, and $C_n' = C_n \sin(c\sqrt{\gamma_n^2+\kappa k_0^2})$. These equations are valid for each $m = 1, 2, 3, \cdots$.

Solution of the Set of Equations

The form of the above infinite sets of simultaneous linear equations differs significantly from those previously solved by the MRCT. However, by modifying certain steps in the original procedure, we are able to extend the MRCT to the above equations and obtain a very accurate approximate solution for the unknown modal coefficients A_n and C_n.

Since the intent is to extend the original MRCT, our development is guided by the original procedure. The key step is the construction of a meromorphic function $f(w)$ with the following properties:

(f:1) $f(w)$ has simple poles at $w = \alpha_n$, and at $w = -\gamma_n$, $n = 1, 2, 3, \cdots$; also at $w = -\alpha_p$,

(f:2) $f(\pm\beta_n) = 0, \quad n = 1, 2, 3, \cdots,$

(f:3)
$$f(\beta_n') + \lambda_n f(-\beta_n') + \xi_n\left[f(\zeta_n) - R_f(-\alpha_p)\left(\frac{1}{\alpha_p+\zeta_n}\right) + \sum_{i=1}^{\infty} R_f(\alpha_i)\left(\frac{1}{\alpha_i-\zeta_n}\right)\right] - \sum_{i=1}^{\infty} R_f(-\gamma_i)\left(\frac{1}{\gamma_i+\beta_n'}+\frac{\lambda_n}{\gamma_i-\beta_n'}\right) = 0, \quad n = 1, 2, 3, \cdots,$$

(f:4)
$$f(-\beta_n') + \lambda_n f(\beta_n') + \xi_n\left[f(-\zeta_n) - R_f(-\alpha_p)\left(\frac{1}{\alpha_p-\zeta_n}\right) + \sum_{i=1}^{\infty} R_f(\alpha_i)\left(\frac{1}{\alpha_i+\zeta_n}\right)\right] - \sum_{i=1}^{\infty} R_f(-\gamma_i)\left(\frac{1}{\gamma_i-\beta_n'}+\frac{\lambda_n}{\gamma_i+\beta_n'}\right) = 0, \quad n = 1, 2, 3, \cdots,$$

(f:5) $f(w) \sim Kw^{-\nu}$ as $|w| \to \infty$ where $\nu > 1$ is given by a physical constraint, the edge condition,

(f:6) the residue of $f(w)$ at $w = -\alpha_p$ is equal to A'.

Properties (f:3) and (f:4) are the significant difference between this work and the original MRCT.

Assuming such a function $f(w)$ can be constructed, consider the following contour integrals in the complex w plane:

$$\frac{1}{2\pi j}\oint_C \left[\frac{f(w)}{w-\beta_m'} + \lambda_m \frac{f(w)}{w+\beta_m'} + \xi_m \frac{f(w)}{w-\zeta_m}\right] dw, \quad m = 1, 2, 3, \cdots$$

$$\frac{1}{2\pi j}\oint_C \left[\frac{f(w)}{w+\beta_m'} + \lambda_m \frac{f(w)}{w-\beta_m'} + \xi_m \frac{f(w)}{w+\zeta_m}\right] dw, \quad m = 1, 2, 3, \cdots$$

$$\frac{1}{2\pi j}\oint_C \frac{f(w)}{w-\beta_m} dw, \quad m = 1, 2, 3, \cdots$$

$$\frac{1}{2\pi j}\oint_C \frac{f(w)}{w+\beta_m} dw, \quad m = 1, 2, 3, \cdots$$

where the contour C is a circle of infinite radius enclosing all the poles and zeros of $f(w)$. The values of the integrals can be shown to be zero by (f:5). Applying the residue theorem together with (f:1)–(f:4) one obtains

$$\sum_{n=1}^{\infty}\left\{R_f(\alpha_n)\left[\frac{1}{\alpha_n-\beta_m'} + \frac{\lambda_m}{\alpha_n+\beta_m'}\right] - \xi_m R_f(-\gamma_n)\left[\frac{1}{\gamma_n+\zeta_m}\right]\right\} - R_f(-\alpha_p)\left[\frac{1}{\alpha_p+\beta_m'} + \frac{\lambda_m}{\alpha_p-\beta_m'}\right] = 0 \quad \text{(3a)}$$

$$\sum_{n=1}^{\infty}\left\{R_f(\alpha_n)\left[\frac{1}{\alpha_n+\beta_m'} + \frac{\lambda_m}{\alpha_n-\beta_m'}\right] - \xi_m R_f(-\gamma_n)\left[\frac{1}{\gamma_n-\zeta_m}\right]\right\} - R_f(-\alpha_p)\left[\frac{1}{\alpha_p-\beta_m'} + \frac{\lambda_m}{\alpha_p+\beta_m'}\right] = 0 \quad \text{(3b)}$$

$$\sum_{n=1}^{\infty}\left\{R_f(\alpha_n)\left[\frac{1}{\alpha_n-\beta_m}\right] - R_f(-\gamma_n)\left[\frac{1}{\gamma_n+\beta_m}\right]\right\} - R_f(-\alpha_p)\left[\frac{1}{\alpha_p+\beta_m}\right] = 0 \quad \text{(3c)}$$

$$\sum_{n=1}^{\infty}\left\{R_f(\alpha_n)\left[\frac{1}{\alpha_n+\beta_m}\right] - R_f(-\gamma_n)\left[\frac{1}{\gamma_n-\beta_m}\right]\right\} - R_f(-\alpha_p)\left[\frac{1}{\alpha_p-\beta_m}\right] = 0 \quad \text{(3d)}$$

for $m = 1, 2, 3, \cdots$. By comparing (3a)–(3d) with (2a)–(2d), it is apparent that the solution to the latter sets of equations is given by

$$A_n' = R_f(\alpha_n), \qquad n = 1, 2, 3, \cdots,$$
$$C_n' = -R_f(-\gamma_n), \qquad n = 1, 2, 3, \cdots.$$

Construction of $f(w)$

Following the reasoning of the original MRCT we are led to write $f(w)$ as

$$f(w) = K_f \exp(Lw) \cdot \frac{\prod(w,\beta)\prod(w,-\beta)\prod(w,\zeta')\prod(w,-\eta)}{(w+\alpha_p)\prod(w,\alpha)\prod(w,-\gamma)} \quad (4)$$

where

$$\prod(w,\zeta') = \prod_{n=1}^{\infty}\left(1 - \frac{w}{\zeta_n'}\right)\exp(cw/n\pi)$$

$$\prod(w,-\eta) = \prod_{n=1}^{\infty}\left(1 + \frac{w}{\eta_n}\right)\exp(-cw/n\pi)$$

and similarly for $\prod(w,\alpha)$, $\prod(w,\beta)$, and $\prod(w,-\gamma)$. The normalization constant K_f is chosen to satisfy (f:6). The sets of zeros $\{\zeta n'\}$ and $\{-\eta_n\}$ are unknown and must be found.

In the original MRCT it was possible to determine the asymptotic behavior of these unknown sets of zeros, then explicitly determine a finite subset of the zeros, and use the known asymptotic expressions for the remainder. However, because of the infinite series appearing in (f:3) and (f:4), this procedure cannot be used in this case. In the original MRCT, the unknown zeros were found to be shifted a slight amount from sets of known zeros. This suggests the use of a perturbation function to add corrections for the shifted zeros to the product representation of the unshifted zeros. Such a function is given by

$$H(w) = \prod(w,\beta')\prod(w,-\beta')\left[1 + \sum_{n=1}^{\infty} S_n\left(\frac{w}{\beta_n'-w}\right) + \sum_{n=1}^{\infty} T_n\left(\frac{w}{\beta_n'+w}\right)\right] \quad (5)$$

where $\{\beta_n'\}$ and $\{-\beta_n'\}$ are the unshifted zeros. The unknown coefficients $\{S_n\}$ and $\{T_n\}$ are the perturbation coefficients to account for the shift in $\{\beta_n'\}$ and $\{-\beta_n'\}$ to $\{\zeta_n'\}$ and $\{-\eta_n\}$, respectively. Note that $H(w)$ has been chosen such that $H(\beta_m') = KS_m$ and $H(-\beta_m') = -KT_m$ for $m = 1, 2, 3, \cdots$, where K is a constant. Also, when w equals zero, $H(0) = 1$, so the essential characteristics of $f(w)$ are maintained.

It would appear that the use of $H(w)$ has not improved the procedure, since it is now necessary to determine two infinite sets of coefficients, where previously two infinite sets of zeros were unknown. However, now the asymptotic behavior of the coefficients $\{S_n\}$ and $\{T_n\}$ can be determined, whereas before we could not determine the behavior of the zeros.

To this end, substitute (5) for $\prod(w, \zeta') \prod(w, -\eta)$ in (4), let $w = \pm\beta_m'$ as $m \to \infty$, use asymptotic values for the known poles and zeros, and require the satisfaction of condition (f:5). After simplifying, the asymptotic behavior as $m \to \infty$ is obtained as

$$\begin{aligned} S_m &\sim m^{1-\nu} \\ T_m &\sim m^{1-\nu} \end{aligned} \tag{6}$$

where the value

$$\nu = 1 + \frac{2}{\pi} \cos^{-1}\left[\frac{\kappa - 1}{2(\kappa + 1)}\right]$$

is known as the edge condition for a right-angle dielectric wedge [4]. It can also be shown that if (6) is true then

$$\begin{aligned} f(\alpha_m) &\sim K_\alpha m^{-\nu} \\ f(-\gamma_m) &\sim K_\gamma m^{-\nu} \end{aligned}$$

for $m \to \infty$ as specifically required by the edge condition.

Knowing the required asymptotic behavior of the perturbation coefficients, it is now possible to rewrite $H(w)$ as

$$\begin{aligned} H(w) = \prod(w, \beta') \prod(w, -\beta') \Bigg[1 &+ \sum_{n=1}^{N_a - 1} S_n \left(\frac{w}{\beta_n' - w}\right) \\ &+ \overline{S} \sum_{n=N_a}^{\infty} \frac{w n^{1-\nu}}{\beta_n' - w} + \sum_{n=1}^{N_b - 1} T_n \left(\frac{w}{\beta_n' + w}\right) \\ &+ \overline{T} \sum_{n=N_b}^{\infty} \frac{w n^{1-\nu}}{\beta_n' + w} \Bigg] \end{aligned} \tag{7}$$

where $\overline{S}$ and $\overline{T}$ are unknown coefficients which account for all the zeros of order equal to or greater than N_a and N_b, respectively. Note that the asymptotic behavior of these higher order coefficients is displayed as $n^{1-\nu}$ and summed.

The original scattering problem has now been transformed into one of finding the unknowns $\{S_n\}$, $\overline{S}$, $\{T_n\}$, and $\overline{T}$. By using (7) for $H(w)$ in (4), and then imposing (f:3) and (f:4), a set of $N = N_a + N_b$ linear simultaneous equations for the N unknown perturbation coefficients is obtained. The infinite series which appear in the expressions pose no significant numerical problems. The series are convergent and can be accurately summed very quickly using appropriate numerical techniques. The solution of this set of linear equations for the unknown perturbation coefficients is readily accomplished. After determining K_f from (f:6) the construction of $f(w)$ is complete.

The need to solve this rather complex set of equations might lead one to question the advantage of extending the MRCT. In practice, as with previous applications of the MRCT, the number of equations which must be solved is quite small compared to the number required by other methods. Usually $N = 10$ or less gives very good results.

Numerical Results for the Semi-Infinite Step

The satisfaction of the equations for the perturbation coefficients serves as a test of the convergence of the numerical calculations. Satisfaction of these equations for $n = 1, 2, 3, \cdots, N$ also is a measure of how well conditions (f:3) and (f:4) on $f(w)$ are met. The satisfaction of the equations, and thus the convergence of the process of calculating the perturbation coefficients, is illustrated by Table I for a typical case.

Table I shows an example for $N = 10$ and $N_a = N_b$. It was found that the final numerical results were insensitive to the ratio N_a/N_b. Therefore, in all the numerical work which follows, N_a was set equal to N_b. The choice $N = 10$ was found to give very accurate results. However, good results were obtained by taking N as few as four.

Of considerable interest in a waveguide discontinuity problem, which has been formulated using the field continuity conditions at some interface, is the satisfaction of these conditions by the numerical results. Also significant in the case of a lossless discontinuity, such as the dielectric step, is the principle of conservation of energy. Consequently, if one defines error criteria based on these fundamental concepts, the criteria so defined may be used to assess the accuracy of the numerical techniques. The mean-square errors in the tangential fields at the $z = 0$ interface are defined as

$$\begin{aligned} \epsilon_E &= \frac{\iint_{\text{Aperture}} |E_y^+ - E_y^-|^2 \, dA}{\iint_{\text{Aperture}} |E_y^{\text{inc}}|^2 \, dA} \\ \epsilon_H &= \frac{\iint_{\text{Aperture}} |H_x^+ - H_x^-|^2 \, dA}{\iint_{\text{Aperture}} |H_x^{\text{inc}}|^2 \, dA} \end{aligned} \tag{8}$$

where E_y^+ and H_x^+ are the tangential fields at $z = 0^+$ in the C region of the guide, E_y^- and H_x^- are taken at $z = 0^-$ in the A region, and E_y^{inc} and H_x^{inc} are the components of the incident wave at $z = 0^-$. The conservation of energy may be evaluated from the expression

$$\epsilon_P = 1 - \frac{P_r + P_t}{P_i} \tag{9}$$

where P_i, P_r, and P_t are the incident, reflected, and transmitted powers, respectively [5].

In this extension of the MRCT the reflection and transmission coefficients are independently determined directly from $f(w)$ by simultaneous calculations. Therefore, the parameters ϵ_E, ϵ_H, ϵ_P defined in (8) and (9) are a valid measure of the accuracy of the solution. Typical

TABLE I
CONVERGENCE CHECK FOR $N_a = N_b = 5$
($\lambda_0 = 1.63a$, $c/a = 0.201$, $\kappa = 9.91$)

n	Residual Value of Equations (f: 3) and (f: 4)
1	$(-0.508+j0.697)\times10^{-17}$
2	$(-0.289+j0.211)\times10^{-17}$
3	$(-0.417+j0.153)\times10^{-18}$
4	$(-0.342+j0.174)\times10^{-18}$
5	$(0.820-j0.631)\times10^{-19}$
6	$(0.363+j0.498)\times10^{-17}$
7	$(0.862+j0.071)\times10^{-18}$
8	$(-0.091+j0.116)\times10^{-18}$
9	$(0.572-j0.172)\times10^{-19}$
10	$(-0.155+j0.074)\times10^{-18}$

TABLE II
ACCURACY OF THE TECHNIQUE

N	Energy Parameter ϵ_P	Mean Square Errors ϵ_E	Mean Square Errors ϵ_H
	($\lambda_0 = 1.63a$, $c/a = 0.101$, $\kappa = 9.91$)		
4	0.485×10^{-4}	0.153×10^{-6}	0.141×10^{-5}
10	0.343×10^{-5}	0.119×10^{-6}	0.106×10^{-5}
16	0.973×10^{-6}	0.119×10^{-6}	0.106×10^{-5}
	($\lambda_0 = 1.2369a$, $c/a = 0.2756$, $\kappa = 2.47$)		
4	0.131×10^{-2}	0.989×10^{-5}	0.199×10^{-4}
8	0.175×10^{-3}	0.252×10^{-6}	0.483×10^{-6}
16	0.228×10^{-4}	0.134×10^{-6}	0.247×10^{-6}

values of ϵ_P, ϵ_E, and ϵ_H are given in Table II as a function of N.

As additional verification, solutions for $c/a = 1$ were obtained for several values of the dielectric constant. The calculated modal coefficients agreed in both magnitude and phase to ± 2 in the fourth significant digit with the well-known exact solution. Further limiting cases are not practical. For example, the equations obtained as $\kappa \rightarrow \infty$ can be solved more efficiently by direct application of the MRCT [2].

One of the thoughts which motivated this study was a concern that previous methods of solving the dielectric step would prove to be inaccurate when applied to the high dielectric constants in use today. Earlier methods have calculated only the first, or at best, the first few mode coefficients. Hence, the accuracy of these methods is critically dependent on the amplitudes of the higher order modes decreasing rapidly. The calculations of this study confirmed the relatively slow rate of decrease in the magnitudes of the reflected mode coefficients for high dielectric constants. However, when one utilizes the extended MRCT to obtain a solution, the significant part of the numerical work is the computation of the perturbation coefficients which determine $f(w)$. Once $f(w)$ is determined the calculation of any number of mode coefficients is easily accomplished by computing the appropriate residues. Thus the MRCT can readily provide a very accurate approximation to the field configuration with little concern for the number of modes required to obtain the desired accuracy.

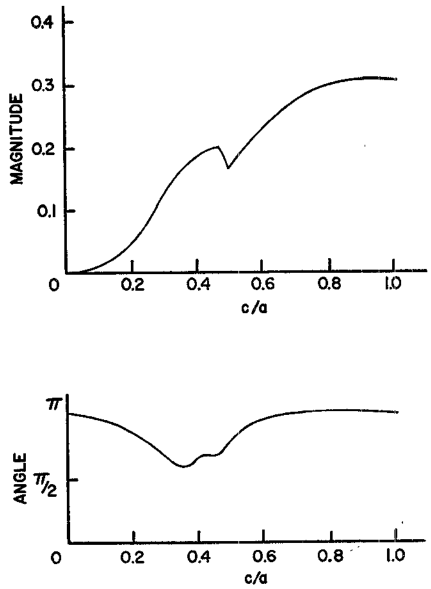

Fig. 2. Reflection coefficient for the semi-infinite step ($\lambda_0 = 0.256a$, $\kappa = 2.56$).

A final example of the computations which may be carried out with the extended MRCT is illustrated in Fig. 2. The break in the curves at $x \cong 0.456$ occurs at the point where two modes begin to propagate in the partially loaded guide. This ability to accurately handle more than one propagating mode without an increase in computational complexity is an important feature of the MRCT. In these calculations, which solved six simultaneous equations, a check of the field continuity at the $z = 0$ interface for each computed point of the curve revealed a match to ± 1 in the third significant digit.

FORMULATION OF THE FINITE DIELECTRIC STEP PROBLEM

The solution of the finite-step problem, which is of considerable practical interest, is carried out in a similar manner. The finite step is formed by truncating the dielectric of Fig. 1(a) at a length $z = d$. To formulate the problem we exploit the symmetry of the finite step about the $z = d/2$ plane by the usual method of considering auxiliary even and odd excitations. For even excitations a magnetic wall at $z = d/2$ has no effect. Likewise, an electric wall may be placed at this plane in the odd excitation case. If both excitations are applied simultaneously, they add for $z < 0$ and cancel for $z > 0$, leaving the desired TE_{p0} excitation which was used previously. Hence, superposition of the solutions to the auxiliary problems depicted in Fig. 3(a) will lead to the solution of the finite dielectric step where the wall at $z = d/2$ is magnetic or electric as the excitation is even or odd, respectively.

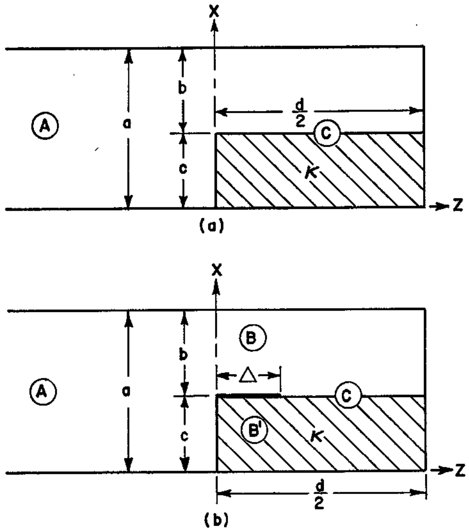

Fig. 3. Equivalent scattering problem with auxiliary geometry.

Again it is convenient to consider the auxiliary geometry shown in Fig. 3(b) which coincides with Fig. 3(a) as $\Delta \to 0$. The mode-matching and limiting procedures used previously result in the following equations, extensions of (2), which must be solved:

$$\sum_{n=1}^{\infty}\left[A_n'\left(\frac{1}{\alpha_n-\beta_m'}+\frac{\lambda_m}{\alpha_n+\beta_m'}\right)+\xi_m C_n'\left(\frac{1}{\gamma_n+\zeta_m}+\frac{t_n}{\gamma_n-\zeta_m}\right)\right]=A'\left(\frac{1}{\alpha_p+\beta_m'}+\frac{\lambda_m}{\alpha_p-\beta_m'}\right) \tag{10a}$$

$$\sum_{n=1}^{\infty}\left[A_n'\left(\frac{1}{\alpha_n+\beta_m'}+\frac{\lambda_m}{\alpha_n-\beta_m'}\right)+\xi_m C_n'\left(\frac{1}{\gamma_n-\zeta_m}+\frac{t_n}{\gamma_n+\zeta_m}\right)\right]=A'\left(\frac{1}{\alpha_p-\beta_m'}+\frac{\lambda_m}{\alpha_p+\beta_m'}\right) \tag{10b}$$

$$\sum_{n=1}^{\infty}\left[\frac{A_n'}{\alpha_n-\beta_m}+C_n'\left(\frac{1}{\gamma_n+\beta_m}+\frac{t_n}{\gamma_n-\beta_m}\right)\right]=\frac{A'}{\alpha_p+\beta_m} \tag{10c}$$

$$\sum_{n=1}^{\infty}\left[\frac{A_n'}{\alpha_n+\beta_m}+C_n'\left(\frac{1}{\gamma_n-\beta_m}+\frac{t_n}{\gamma_n+\beta_m}\right)\right]=\frac{A'}{\alpha_p-\beta_m} \tag{10d}$$

where $m=1, 2, 3, \cdots$.

These equations differ from the previous ones for the semi-infinite step by virtue of the terms containing t_n, which is given by $\exp(-\gamma_n d/2)$. The unknowns $\{A_n\}$ and $\{C_n\}$ correspond to the scattering coefficients in regions A and C of Fig. 3(a), respectively. The above equations are for the electric-wall case. To obtain the magnetic-wall equations, the sign of t_n must be reversed.

The solution of (10) entails a further extension of the extended MRCT. Following a development analogous to that used previously, we consider a function $h(w)$ similar to $f(w)$. The details of the construction of $h(w)$ will not be given, as they are readily available [6]. As before, the unknowns $\{A_n\}$ and $\{C_n\}$ will be given by appropriate residues of $h(w)$. The expression for $h(w)$ can be shown to be

$$h(w)=K_h\frac{\prod(w,\beta)\prod(w,-\beta)\prod(w,\beta')\prod(w,-\beta')}{\left(1+\frac{w}{\alpha_p}\right)\prod(w,\alpha)\prod(w,-\gamma)}\cdot\left[1+\sum_{n=1}^{N_a-1}S_n\left(\frac{w}{\beta_n'-w}\right)+\overline{S}\sum_{n=N_a}^{\infty}w\left(\frac{n^{1-\nu}}{\beta_n'-w}\right)+\sum_{n=1}^{N_b-1}T_n\left(\frac{w}{\beta_n'+w}\right)+\overline{T}\sum_{n=N_b}^{\infty}w\left(\frac{n^{1-\nu}}{\beta_n'+w}\right)+\sum_{n=1}^{\infty}V_n\left(\frac{w}{\gamma_n-w}\right)\right] \tag{11}$$

where the $\{S_n\}$, $\{T_n\}$, $\overline{S}$, and $\overline{T}$ are the perturbation coefficients previously defined. The asymptotic behavior of the newly introduced unknown coefficients $\{V_n\}$, whose use was suggested by Itoh and Mittra [7], can be determined *a priori*. It can be shown that $V_n \sim n^{-1-\nu}\exp(-n\pi d/2a)$ as $n\to\infty$. Thus the series containing V_n in (11) can be truncated at a few terms, say M, with negligible error.

The unknown coefficients in $h(w)$ are determined from conditions similar to $(f\!:\!3)$ and $(f\!:\!4)$ together with an additional restraint which must be imposed on the residues of $h(w)$. The resulting set of $N+M$ simultaneous linear equations is then solved once for the electric-wall case and again for the magnetic-wall case. The majority of the numerical calculations need not be repeated, however, so the method is numerically efficient. The resulting scattering coefficients for the two auxiliary problems are then combined to yield the total reflected and transmitted fields for the finite step.

Numerical Results for the Finite Step

This further extension of the extended MRCT gives numerical results comparable in accuracy to those obtained for the semi-infinite dielectric step. The numerical procedure for the calculation of the unknown coefficients

TABLE III
ACCURACY OF THE SOLUTION OF THE FINITE DIELECTRIC STEP
($\lambda_0=1.24a$, $c/a=0.101$, $\kappa=6.73$, $d=\lambda_0$)

N	M	Energy Parameter ϵ_P	Mean Square Errors ϵ_E	ϵ_H
Electric Wall Case				
4	1	-0.273×10^{-3}	0.370×10^{-2}	0.182×10^{-2}
10	4	-0.176×10^{-4}	0.430×10^{-6}	0.622×10^{-6}
16	4	-0.554×10^{-5}	0.432×10^{-6}	0.599×10^{-6}
Magnetic Wall Case				
4	1	-0.654×10^{-4}	0.242×10^{-3}	0.109×10^{-3}
10	4	-0.127×10^{-4}	0.323×10^{-6}	0.264×10^{-5}
16	4	-0.401×10^{-5}	0.324×10^{-6}	0.264×10^{-5}

Total Solution

N	M	Energy Parameter ϵ_P
4	1	0.104×10^{-3}
10	4	-0.152×10^{-5}
16	4	-0.478×10^{-5}

in the function $h(w)$ exhibited convergence properties equivalent to those previously obtained.

The tangential field match was checked for each case that was investigated. Some typical results, as measured by the mean-square error in the tangential field match, are presented in Table III. The energy parameter for the final solution is also presented. A point-by-point direct matching of the magnitudes of the tangential electromagnetic fields revealed a match to three significant digits for $N=16$ and $N=10$, with $M=4$ for both the electric- and magnetic-wall cases. For $N=4$ and $M=2$ the fields matched to ± 3 in the second significant digit. With $N=10$, choosing $M=2$ rather than $M=4$ had negligible effect on the accuracy of the results. This was expected because of the exponential behavior of the coefficients V_n. The choice of $N=10$ and $M=2$ was found to provide the best compromise between minimum computation time and optimum accuracy.

The result of varying the thickness of the dielectric while maintaining a given permittivity and step length is shown in Fig. 4. The increased reflection at $c/a \cong 0.15$ had a counterpart in the semi-infinite dielectric step (Fig. 2) and corresponds to the beginning of double-mode propagation in the dielectric-filled waveguide. The explanation for the return to a small reflection coefficient for $0.20<c/a<0.25$ undoubtedly lies in the particular configuration of the propagating fields and the resultant coupling through the finite step.

CONCLUSIONS

The extensions of the MRCT reported here provide very accurate solutions to the equations obtained for the dielectric-step problems, even when the permittivity or frequency becomes very large. The method is numerically efficient as matrix size is typically reduced by a factor of 10 compared to previous methods. The convergence of the solution is easily confirmed. An additional significant advantage is that the uniqueness is *a priori* guaranteed by requiring the satisfaction of the edge condition as a step in the procedure.

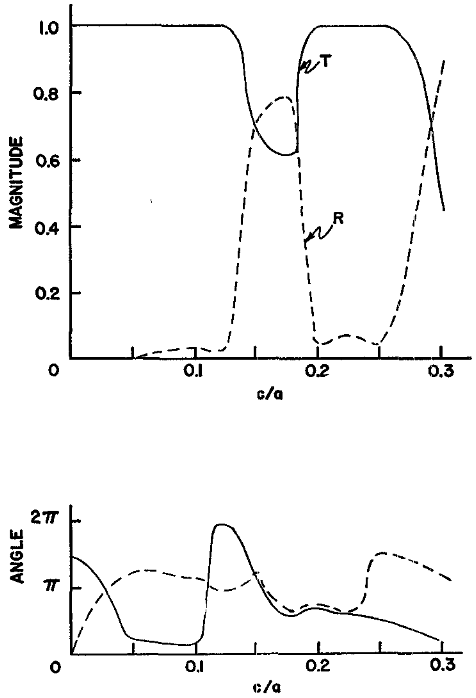

Fig. 4. Reflection and transmission coefficients versus step height for the finite step ($\lambda_0=1.5376a$, $\kappa=6.73$, $d=2\lambda_0$).

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